EXACT SIMULATION OF EXTRINSIC STRESS-RELEASE PROCESSES

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Abstract

We present a new and straightforward algorithm that simulates exact sample paths for a generalized stress-release process. The computation of the exact law of the joint inter-arrival times is detailed and used to derive this algorithm. Furthermore, the martingale generator of the process is derived, and induces theoretical moments which generalize some results of [3] and are used to demonstrate the validity of our simulation algorithm.

Keywords: Stress-release process; inhibitory point process

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1. Introduction

Stress-release processes are a class of point processes, the first of which were self-correcting processes [7]. Intuitively, a process is self-correcting if the occurrence of past points inhibits the occurrence of future points. The stress-release processes are a generalization of self-correcting processes and were introduced in a series of papers by Vere-Jones and others [24, 25] as well as extensions to coupled stress-release processes [11, 12, 18] and further developments [1, 2].

In this study we work with a *generalization* of the stress-release process which includes an exogenous point process term whose values upon arrival are modeled by a *positive* real-valued random variable. We call our model the *extrinsic stress-release processes*.

We present a new formula for the law of the joint inter-arrival times for extrinsic stress-release processes. As a natural consequence, an exact simulation algorithm is then proposed which gives an alternative method to generating sample paths relative to standard methods [9]. Our exact simulation algorithm naturally extends the results of [22] as a special case. The

extension of our model is motivated by the influence of exogenous geophysical data on earthquake occurrence (see e.g. [15] and [26]). Point process models of this kind are typically used to describe the evolution of stochastic phenomena in earthquake modeling, and it is important to be able to simulate them for reliable predictions of damage due to a range of earthquake scenarios. Thinning algorithms [14] have been successfully employed to simulate a wide range of point processes, such as inhomogeneous point processes (see [5, pages 270–271] and [23]), or Hawkes processes [19]. Indeed, the same idea can be applied to the generalized stress-release process proposed here. In this paper, simulation of the extrinsic stress-release process by our exact algorithm will be compared with the standard thinning algorithm.

Finally, we present the infinitesimal generator for extrinsic stress-release processes. This generator is intimately linked to the martingale problem, which is used to characterize the weak solutions of partial integro-differential equations [10], and it allows us to derive the theoretical reciprocal moments of the intensity function. In Section 6 these reciprocal moments are used to demonstrate the correctness of our simulation algorithms. Basic notions and results in stochastic calculus are taken as prerequisites throughout the present text (see e.g. [17]).

2. Extrinsic stress-release model

At the base of everything is some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We assume that $\mathcal{F}_0$ is trivial, and the filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ fulfills the usual conditions and is generated by a point process $N(\cdot)$ on $\mathbb{R}_+$ where $0 < T_1 < T_2 < \cdots$ denote the occurrence times of the events. Let $N_t = \sharp\{T_i : 0 < T_i \leq t\}$ be the number of the occurrence points in the time interval $(0, t]$ with $N_0 = 0$. Furthermore, we let $N'_t = \sharp\{T'_j : 0 < T'_j \leq t\}$ be a Poisson process on $\mathbb{R}_+$, with arrival times $0 < T'_1 < T'_2 < \cdots$ endowed with intensity ρ, which is independent of N_t, with $N'_0 = 0$.

Definition 2.1. The proposed extrinsic stress-release process $N(\cdot)$ is a point process on $\mathbb{R}_+$ with conditional intensity function given by

$$\lambda_t := \lambda(t \mid \mathcal{F}_t) = \lambda_0 \exp\left(\beta t - S_t - S'_t\right), \quad t \geq 0, \tag{2.1}$$

where $S_t = \sum_{i\,:\,T_i < t} X_i$ and $S'_t = \sum_{j\,:\,T'_j < t} Y_j$ are the compound point process and compound Poisson process, respectively. The X_i and Y_j are i.i.d. positive random variables, with distribution functions F_X and F_Y respectively, and the stress accumulation rate is the constant $\beta > 0$.

Between jumps, λ_t increases exponentially with a positive rate of $\beta > 0$. Jumps are downward multiplicative factors of size $e^{-X_i} < 1$ for a self-arrival at time T_i, or of size $e^{-Y_j} < 1$ for an external arrival at time T'_j. When a self-arrival occurs at time T_j, N_t increases by one, hence (N_t, λ_t) is a Markov process. Instead of separating the self-arrivals of N_t and the external arrivals of N'_t, it is sometimes convenient to consider all the arrivals indiscriminately. As such, we label the kth arrival as T_k°, and it can correspond either to some self-arrival T_i or some external arrival T'_j. See Figure 1 for an example realization of the extrinsic stress-release process, with the effects of the self-arrivals and external arrivals on the conditional intensity function λ_t highlighted.

Remark 2.1. Our proposed extrinsic stress-release process differs slightly from the coupled stress-release model of [11]. Their equivalent of X_i and Y_j from Definition 2.1 are not unobserved random variables: they are deterministic functions of the observed earthquake magnitudes. In our proposed model, we allow for these quantities to take any i.i.d. random
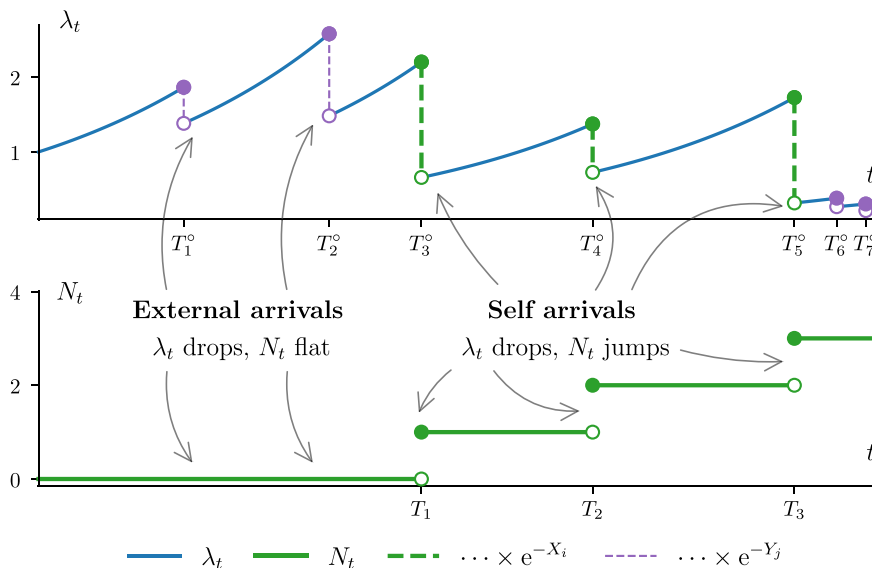
FIGURE 1. An example realization of an extrinsic stress-release process, with $\lambda_0 = 1$, $\beta = 1.5$, $\rho = 2$, and $X_i \sim \mathsf{Exp}(1)$ and $Y_j \sim \mathsf{Exp}(2)$. Note that N is càdlàg while λ is càglàd.

variables that are positive and unobserved, so our formulation generalizes theirs. Unlike ours, they allow for model parameters c and c' in the exponent of equation (2.1) of the form

$$-cS_t - c'S'_t,$$

where c and c' can take either negative or positive values, thereby allowing for both damping and excitation. We only consider the inhibitory regime, i.e. $c = c' = 1$, so in this case their formulation for general c and c' subsumes ours. In either formulation, little or no work has appeared on exact simulation strategies for the coupled stress-release model. We also add some new aspects to the computation of explicit generators, facilitating moment computations. For other theoretical and stationary moment calculations without the exogenous term S', see [3], [16], [20], and [21].

3. The law of joint inter-arrival times

In this section we present the explicit law of the joint inter-arrival times for extrinsic stress-release processes. This terminology, 'joint inter-arrival time', refers to the time between each of T_k° arrivals (defined in Section 2). Itô's formula [8] splits λ_t into continuous and jump components:

$$\lambda_t = \lambda_0 + \int_0^t \beta \lambda_s \mathrm{d}s + \sum_{i:T_i \leq t} \lambda_{T_i} \left(\mathrm{e}^{-X_i} - 1 \right) + \sum_{j:T'_j \leq t} \lambda_{T'_j} \left(\mathrm{e}^{-Y_j} - 1 \right).$$

Between consecutive jumps the λ process evolves according its continuous part. In particular, conditioned on T_k° and $\lambda_{T_k^{\circ+}}$, we have

$$\lambda_t = \lambda_{T_k^{\circ+}} \exp \left(\beta(t - T_k^\circ) \right) \quad \text{for } t \in (T_k^\circ, T_{k+1}^\circ). \tag{3.1}$$

The intensity of the T_k° arrivals is the combination of the T_i and T_j' arrival intensities $\lambda_t + \rho$. Let the kth joint inter-arrival time be denoted by $\tau_k := T_k^\circ - T_{k-1}^\circ$ with cumulative density function F_{τ_k}. With (3.1), we can simplify the point process relation

$$F_{\tau_{k+1}}(t) = 1 - \exp\left(-\int_{0^+}^t (\lambda_{T_k^\circ + s} + \rho)\mathrm{d}s\right) = 1 - \exp\left(-\frac{\lambda_{T_k^{\circ+}}}{\beta}(\mathrm{e}^{\beta t} - 1)\right)\mathrm{e}^{-\rho t}, \qquad (3.2)$$

which is the law of the joint inter-arrival times.

4. Simulation methods

The law of the joint inter-arrival times in (3.2) can be used to derive a simulation method for extrinsic stress-release processes. To simulate we need to (i) generate T_k° joint inter-arrival times, and (ii) be able to attribute each arrival as being either a self-arrival from N_t or an external arrival from N_t'. By the inverse probability integral transform, we have

$$\tau_{k+1} \overset{\mathcal{D}}{=} F_{\tau_{k+1}}^{-1}(U), \quad U \sim \mathsf{U}[0, 1],$$

where $\overset{\mathcal{D}}{=}$ denotes equality in distribution. The inverse $F_{\tau_{k+1}}^{-1}$ does have an analytic solution (which is somewhat rare) in terms of the Lambert W-function, so we can generate joint inter-arrival times by the inverse transform method. However, the Lambert W-function is relatively slow in many software packages, and this calculation does not perform the second attribution step. A faster alternative, which solves both problems at once, is to use the *composition method*.

4.1. Exact simulation of stress-release model

The composition method [6, Section VI.2.3] simulates τ_{k+1} from two simpler independent random variables $\tau_{k+1}^{(1)}$ and $\tau_{k+1}^{(2)}$ by taking

$$\tau_{k+1} \overset{\mathcal{D}}{=} \tau_{k+1}^{(1)} \wedge \tau_{k+1}^{(2)},$$

where the notation $\tau_{k+1}^{(1)} \wedge \tau_{k+1}^{(2)}$ is simply shorthand for $\min\left\{\tau_{k+1}^{(1)}, \tau_{k+1}^{(2)}\right\}$. One way to satisfy this relation is to choose

$$\mathbb{P}\left(\tau_{k+1}^{(1)} > s\right) = \exp\left(-\lambda_{T_k^{\circ+}}\beta^{-1}\left(\mathrm{e}^{\beta s} - 1\right)\right) \quad \text{and} \quad \mathbb{P}\left(\tau_{k+1}^{(2)} > s\right) = \mathrm{e}^{-\rho s},$$

so

$$\tau_{k+1}^{(1)} \overset{\mathcal{D}}{=} \frac{1}{\beta} \log\left(1 - \frac{\beta}{\lambda_{T_k^\circ}} \log(U_1)\right), \quad \tau_{k+1}^{(2)} \overset{\mathcal{D}}{=} -\frac{1}{\rho} \log(U_2), \quad U_1, U_2 \sim \mathsf{U}[0, 1]. \qquad (4.1)$$

This is the key step in the composition algorithm, presented in full in Algorithm 1.1.

4.2. Simulation by thinning

Extrinsic stress-release processes can also be simulated via the *thinning algorithm*. The basic idea in this method is to generate a point process that has more arrivals than the model dictates, then probabilistically remove the excess points. The result can be computationally inefficient, and we compare the runtime of the thinning and composition simulation methods in Section 6.

Algorithm 1.1: Generate an extrinsic stress-release process by composition.

Input: start intensity λ_0, stress rate β, external arrival rate ρ, jump size distributions F_X and F_Y, end time T

1 **begin**
2   Initialize $T_0^\circ \leftarrow 0$, $\epsilon \leftarrow 10^{-10}$ or similar, $\lambda_\epsilon \leftarrow \lambda_0$, $i \leftarrow 0$, $j \leftarrow 0$, $k \leftarrow 0$;
3   Simulate $\tau_{k+1}^{(1)}$ and $\tau_{k+1}^{(2)}$ via equation (4.1) and let $T_{k+1}^\circ \leftarrow T_k^\circ + \tau_{k+1}^{(1)} \wedge \tau_{k+1}^{(2)}$;
4   **if** $T_{k+1}^\circ > T$ **then**
5   **return** self-arrivals $\{T_1, \ldots, T_i\}$ and, if desired, external arrivals $\{T_1', \ldots, T_j'\}$
6   **end**
7   Calculate $\lambda_{T_{k+1}^\circ} \leftarrow \lambda_{T_k^\circ + \epsilon} \exp(\beta(T_{k+1}^\circ - T_k^\circ))$ by (3.1) ;
8   **if** $\tau_{k+1}^{(1)} < \tau_{k+1}^{(2)}$ **then**
9   Update $i \leftarrow i + 1$ and $T_i \leftarrow t + \tau_{k+1}^{(1)}$;   ▷ `Self-arrival`
10   Simulate $X_i \sim F_X$ and update $\lambda_{T_{k+1}^\circ + \epsilon} \leftarrow \lambda_{T_{k+1}^\circ} \mathrm{e}^{-X_i}$;
11   **else**
12   Update $j \leftarrow j + 1$ and $T_j' \leftarrow t + \tau_{k+1}^{(2)}$;   ▷ `External arrival`
13   Simulate $Y_j \sim F_Y$ and update $\lambda_{T_{k+1}^\circ + \epsilon} \leftarrow \lambda_{T_{k+1}^\circ} \mathrm{e}^{-Y_j}$;
14   **end**
15   Update $k \leftarrow k + 1$ and go to line 3 ;
16 **end**

The first step in the thinning algorithm is to generate the N_t' and S_t' processes. The self-arrivals are then generated conditional on these external arrivals. Each self-arrival is generated sequentially and requires a local upper bound on the intensity process. If we know S_t' for all $t \in \mathbb{R}_+$, we obviously have

$$\lambda_t \leq \lambda_0 \exp(\beta t - S_t'), \quad t \in \mathbb{R}_+,$$

though as t increases this becomes an extremely loose bound. However, if we also know the process S_t up until time τ, then

$$\overline{\lambda}_{t|\tau} := \lambda_0 \exp(\beta t - S_{t \wedge \tau} - S_t'), \quad t \in \mathbb{R}_+, \tag{4.2}$$

is a much tighter upper bound on the intensity, at least for $t \in (\tau, \tau + \Delta)$ for moderately small Δ. Figure 2 shows some example realizations of (4.2).

With this definition, we can describe the thinning algorithm for the generalized stress-release process in Algorithm 1.2. In particular, line 5 of the algorithm uses (4.2) to find an upper bound of λ_t over a small region $t \in (\tau, \tau + \Delta]$; these maximum values are not too tedious to find, as they occur either at the end time $\tau + \Delta$ or at one of the external arrival times T_j' that arrives inside the region.
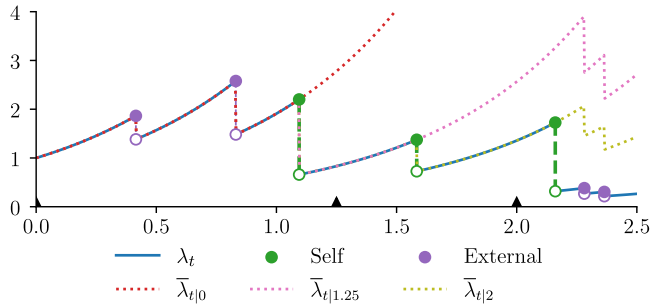
FIGURE 2. Example $\overline{\lambda}_{t|\tau}$ upper bounds on the intensity function λ_t. This is the same realization of the generalized stress-release process from Figure 1.

Algorithm 1.2: Generate an extrinsic stress-release process by thinning.

Input: start intensity λ_0, stress rate β, external arrival rate ρ, jump size distributions F_X and F_Y, end time T, step size Δ

1 **begin**
2 | Simulate the external arrivals $\{T'_1, \ldots, T'_{N'_T}\}$ at rate ρ by standard Poisson process methods ;
3 | Simulate i.i.d. external jump sizes $\{Y_1, \ldots, Y_{N'_T}\}$ from F_Y and construct the $(S'_t)_{t \in [0,T]}$
   | process ;
4 | Initialize $i = 0$, $t = 0$, $\epsilon \leftarrow 10^{-10}$ or similar ;
5 | Set M to be the maximum value of $\overline{\lambda}_{s|t+\epsilon}$, cf. (4.2), over $s \in (t, t + \Delta]$;
6 | Generate a proposal self-arrival $T^* = t + E$ where $E \sim \mathsf{Exp}(M)$;
7 | **if** $T^* > T$ **then**
8 | | **return** self-arrivals $\{T_1, \ldots, T_i\}$
9 | **else if** $T^* > t + \Delta$ **then**
10 | | Reject the proposal, set $t \leftarrow t + \Delta$, go to line 5 ;
11 | **end**
12 | Sample $U \sim \mathsf{U}[0, 1]$;
13 | **if** $U \le \overline{\lambda}_{T^*|t} / M$ **then**
14 | | Accept the proposal: $i \leftarrow i + 1$, $T_i \leftarrow T^*$;
15 | | Simulate a jump size $X_i \sim F_X$ and update $\lambda_{T_i+\epsilon} \leftarrow \lambda_{T_i} \mathrm{e}^{-X_i}$;
16 | | Set $t \leftarrow T^*$, go to line 5 ;
17 | **else**
18 | | Reject the proposal: $t \leftarrow t + \Delta$, go to line 5 ;
19 | **end**
20 **end**

---

5. The generator

In this section we derive the explicit form of the infinitesimal generator for our process. With this we are able to find reciprocal moments, which are then used to confirm the validity of our simulation algorithm.

5.1. Constructing the infinitesimal generator

Let us introduce the integro-differential operator $\mathcal{L}^{sr}$ of our extrinsic stress-release process (λ_t, N_t, t), which acts on a function $f(\lambda, n, t)$ within its domain $\Omega(\mathcal{L}^{sr})$ as follows:

$$
\mathcal{L}^{sr}f := \frac{\partial f}{\partial t} + \beta\lambda\frac{\partial f}{\partial t} + \lambda\int_{\mathbb{R}} [f(\lambda e^{-x}, n+1, t) - f(\lambda, n, t)]\, F_X(\mathrm{d}x)
$$

$$
+ \rho\int_{\mathbb{R}} [f(\lambda e^{-y}, n, t) - f(\lambda, n, t)]\, F_Y(\mathrm{d}y). \tag{5.1}
$$

From Propositions II.1.16 & II.1.15 in [8], the conditional intensity function in equation (2.1) can be recast as

$$
\lambda_t = \lambda_0 \exp\left(\beta t - \int_0^t \int_{\mathbb{R}} x\mu(\mathrm{d}x, \mathrm{d}s) - \int_0^t \int_{\mathbb{R}} y\mu'(\mathrm{d}y, \mathrm{d}s)\right),
$$

where μ and μ' are the jump measures associated with S and S', respectively. Their associated predictable compensators are $\nu(\mathrm{d}x, \mathrm{d}t) = F_X(\mathrm{d}x)\lambda_t\mathrm{d}t$ and $\nu'(\mathrm{d}y, \mathrm{d}t) = F_Y(\mathrm{d}y)\rho\mathrm{d}t$. We now state the following result.

Proposition 5.1. *Let the integro-differential operator for our stress-release process be defined as in equation (5.1). Then, for each $t \in \mathbb{R}_+$, we obtain*

$$
\mathbb{E}[f(\lambda_t, N_t, t)] = f(0, N_0, \lambda_0) + \mathbb{E}\left[\int_0^t \mathcal{L}^{sr}f(\lambda_s, N_s, s)\mathrm{d}s\right]
$$

if the following f-integrability conditions hold:

$$
\mathbb{E}\left[\int_0^t \lambda_s\mathrm{d}s \int_{\mathbb{R}} [f(\lambda_{s^+}, N_s, s) - f(\lambda_s, N_{s^-}, s)]^2\, F_X(\mathrm{d}x)\right] < \infty \tag{5.2}
$$

and

$$
\mathbb{E}\left[\int_0^t \rho\mathrm{d}s \int_{\mathbb{R}} [f(\lambda_{s^+}, N_s, s) - f(\lambda_s, N_{s^-}, s)]^2\, F_Y(\mathrm{d}y)\right] < \infty. \tag{5.3}
$$

Moreover, f satisfies the following partial integro-differential equation:

$$
\frac{\partial f}{\partial t} + \beta\lambda\frac{\partial f}{\partial \lambda} + \lambda\int_{\mathbb{R}} [f(\lambda e^{-x}, n+1, t) - f(\lambda, n, t)]F_X(\mathrm{d}x)
$$

$$
+ \rho\int_{\mathbb{R}} [f(\lambda e^{-y}, n, t) - f(\lambda, n, t)]F_Y(\mathrm{d}y) = 0. \tag{5.4}
$$

Proof. First note that λ_t can be recast as

$$
\lambda_t = \lambda_0 + \int_0^t \beta\lambda_s\mathrm{d}s + \int_0^t \int_{\mathbb{R}} \lambda_s(e^{-x} - 1)\mu(\mathrm{d}x, \mathrm{d}s) + \int_0^t \int_{\mathbb{R}} \lambda_s(e^{-y} - 1)\mu'(\mathrm{d}y, \mathrm{d}s).
$$

Invoking Itô's formula (see [13, Chapter VI]) on the arbitrary function $f(\lambda_t, N_t, t)$ yields

$$
f(\lambda_t, N_t, t) = f(\lambda_0, N_0, 0) + \int_0^t \frac{\partial f}{\partial s}\mathrm{d}s + \int_0^t \frac{\partial f}{\partial \lambda}\mathrm{d}\lambda_s^c + \sum_{0 < s \le t} [f(\lambda_{s^+}, N_s, s) - f(\lambda_s, N_{s^-}, s)],
$$

where λ^c denotes the continuous part of the semimartingale. We can write

$$\sum_{0<s\leq t} [f(\lambda_{s^+}, N_s, s) - f(\lambda_s, N_{s^-}, s)]$$

$$= \sum_{0<s\leq t} [f(\lambda_{s^+}, N_s, s) - f(\lambda_s, N_{s^-}, s)] \cdot \Delta N_s$$

$$+ \sum_{0<s\leq t} [f(\lambda_{s^+}, N_s, s) - f(\lambda_s, N_{s^-}, s)] \cdot \Delta N'_s$$

$$= Q_t(f) + \int_0^t \int_{\mathbb{R}} [f(\lambda_{s^+}, N_s, s) - f(\lambda_s, N_{s^-}, s)] \, \nu(\mathrm{d}x, \mathrm{d}s)$$

$$+ Q'_t(f) + \int_0^t \int_{\mathbb{R}} [f(\lambda_{s^+}, N_s, s) - f(\lambda_s, N_{s^-}, s)] \, \nu'(\mathrm{d}x, \mathrm{d}s)$$

with $Q_\cdot(f)$ and $Q'_\cdot(f)$ being processes defined by

$$Q_\cdot(f) := \int_0^\cdot \int_{\mathbb{R}} [f(\lambda_{s^+}, N_s, s) - f(\lambda_s, N_{s^-}, s)](\mu - \nu)(\mathrm{d}x, \mathrm{d}s)$$

and

$$Q'_\cdot(f) := \int_0^\cdot \int_{\mathbb{R}} [f(\lambda_{s^+}, N_s, s) - f(\lambda_s, N_{s^-}, s)](\mu' - \nu')(\mathrm{d}x, \mathrm{d}s),$$

respectively. The integrability conditions in equations (5.2) and (5.3) guarantee that $Q(f)$ and $Q'(f)$ are square-integrable martingales [4, Theorem VIII of Chapter II]. Hence we conclude that the process $(f(\lambda_t, N_t, t))_{t\in\mathbb{R}_+}$ is a special semimartingale [17] which can be decomposed into a martingale and a predictable finite variation process

$$f(\lambda_t, N_t, t) - f(\lambda_0, N_0, 0) = Q_t(f) + Q'_t(f) + \int_0^t \mathcal{L}^{\mathrm{sr}} f(\lambda_s, N_s, s) \, \mathrm{d}s. \qquad (5.5)$$

Since $Q(f)$ and $Q'(f)$ are square-integrable martingales, the process defined by

$$\left(f(\lambda_t, N_t, t) - f(\lambda_0, N_0, 0) - \int_0^t \mathcal{L}^{\mathrm{sr}} f(\lambda_s, N_s, s) \, \mathrm{d}s \right)_{t\in\mathbb{R}_+}$$

is also a square-integrable martingale. Taking the expected value of both sides of equation (5.5) yields the result. For the subsequent expression, first define $T > t$ and $g_t := \mathbb{E}[h(\lambda_T, N_T, T) \mid t, \lambda_t = \lambda, N_t = n]$ for some function h such that g satisfies the g-integrability conditions (5.2) and (5.3). Then, by construction, g_t is a martingale. By similar arguments, we see that $g - Q(g) - Q'(g)$ is a square-integrable martingale, but $g - Q(g) - Q'(g) = \int_0^\cdot \mathcal{L}^{\mathrm{sr}} g \mathrm{d}s$ is also a continuous process with finite variation. It must therefore be a continuous martingale with finite variation [8, Corollary I-3.16]; hence we must have $\mathcal{L}^{\mathrm{sr}} g = 0$ $\mathbb{P}$-almost surely, which yields the partial integro-differential equation in (5.4). $\qquad\square$

5.2. Reciprocal moments

Consistent with the observation in [21], we are unable to find moments of λ_t but we can find moments of its reciprocal. Throughout Sections 5.2 and 5.3 we assume that $\lambda_0 = 1$, though the same arguments can be made in the general λ_0 case.

Lemma 5.1. *Let $m_1^S := \int (e^x - 1) F_X(dx)$ and $m_1^E := \int (e^y - 1) F_Y(dy)$. We assume that $\beta > \rho m_1^E$ and $\lambda_0 = 1$. Then the expectation of λ_t^{-1} is given by*

$$\mathbb{E}[\lambda_t^{-1}] = e^{-\psi_1 t} + \frac{m_1^S}{\psi_1}(1 - e^{-\psi_1 t}), \tag{5.6}$$

where $\psi_1 := \beta - \rho m_1^E$.

Proof. From Proposition 5.1, we have for $f \in \Omega(\mathcal{L}^{\mathrm{sr}})$ that

$$f(\lambda_t, N_t, t) - f(\lambda_0, N_0, 0) - \int \mathcal{L}^{\mathrm{sr}} f(\lambda_s, N_s, s) ds$$

is an $\mathcal{F}$-martingale. Setting $f = \lambda^{-1}$ in the generator yields

$$\mathcal{L}^{\mathrm{sr}}(\lambda^{-1}) = -\beta \lambda^{-1} + m_1^S + \rho \lambda^{-1} m_1^E$$

and

$$\mathbb{E}\left[\lambda_t^{-1} - \lambda_0^{-1} - \int_0^t \mathcal{L}^{\mathrm{sr}}(\lambda_s^{-1}) ds\right] = 0.$$

Differentiating $\theta_1(t) := \mathbb{E}[\lambda_t^{-1}]$ with respect to t yields the non-linear inhomogeneous ODE

$$\theta_1' + \psi_1 \theta_1 = m_1^S, \quad \theta_1(0) = 1,$$

whose solution is given in equation (5.6). $\qquad \square$

By a similar token, and setting $f = \lambda^{-2}$, we state the following.

Lemma 5.2. *Let*

$$m_2^S := \int \left(e^{2x} - 1\right) F_X(dx), \quad m_2^E := \int \left(e^{2x} - 1\right) F_Y(dy).$$

We assume that $\beta > \rho m_1^E$, $2\beta > \rho m_2^E$, and $\lambda_0 = 1$. Then the expectation of λ_t^{-2} is given by

$$\mathbb{E}[\lambda_t^{-2}] = e^{-\psi_2 t} + m_2^S \left\{ \frac{e^{-\psi_1 t} - e^{-\psi_2 t}}{\psi_2 - \psi_1} + \frac{m_1^S}{\psi_1} \left[\left(\frac{1}{\psi_2} - \frac{e^{-\psi_1 t}}{\psi_2 - \psi_1} \right) - e^{-\psi_2 t} \left(\frac{1}{\psi_2} - \frac{1}{\psi_2 - \psi_1} \right) \right] \right\}, \tag{5.7}$$

where $\psi_2 := 2\beta - \rho m_2^E$.

We end this section by giving some recursive relationships related to the inverse moments of our process. Let $\mathbb{N}$ be the set of natural numbers and let $k \in \mathbb{N}$, $m_k^S := \int (e^{kx} - 1) F_X(dx)$, $m_k^E := \int (e^{ky} - 1) F_Y(dy)$, and $\psi_k := k\beta - \rho m_k^E$. We further assume that $k\beta > \rho m_k^E$. Then the generator for the function $f = \lambda^{-k}$ is readily computed as follows:

$$\mathcal{L}^{\mathrm{sr}}(\lambda^{-k}) = m_k^S \lambda^{k-1} - \psi_k \lambda^k.$$

By the martingale property we have that

$$\mathbb{E}\left[\lambda_t^{-k} - \lambda_0^{-k} - \int_0^t \mathcal{L}^{\mathrm{sr}}(\lambda_s^{-k})\mathrm{d}s\right] = 0. \tag{5.8}$$

Define the quantity $\theta_k(t) := \mathbb{E}[\lambda_t^{-k}]$, and differentiating equation (5.8) with respect to t, we arrive at the recursive non-linear inhomogeneous ODE

$$\theta_k'(t) + \psi_k \theta_k(t) = m_k^S \theta_{k-1}(t),$$

where $\theta_0 \equiv 1$, which can be solved in a recursive fashion to obtain further reciprocal moments.

5.3. Covariance process

We provide an expression for the mean of the product of two reciprocals of intensities for our process. This can be used to compute the covariance process. For $s < t$, it holds true that

$$\mathbb{E}\left[\lambda_t^{-1}\lambda_s^{-1}\right] = \mathbb{E}\left[\mathbb{E}\left[\lambda_t^{-1}\lambda_s^{-1} \mid \lambda_s^{-1}\right]\right]$$

$$= \mathrm{e}^{-\psi_1(t-s)}\mathbb{E}\left[\lambda_s^{-2}\right] + \frac{m_1^S}{\psi_1}(1 - \mathrm{e}^{-\psi_1(t-s)})\mathbb{E}\left[\lambda_s^{-1}\right]. \tag{5.9}$$

To see why this is true, note that the inner expectation can be computed as follows:

$$\mathbb{E}\left[\lambda_t^{-1}\lambda_s^{-1}|\lambda_s^{-1}\right] = \lambda_s^{-1}\mathbb{E}\left[\lambda_t^{-1}|\lambda_s^{-1}\right] = \lambda_s^{-2}\mathrm{e}^{-\psi_1(t-s)} + \lambda_s^{-1}\frac{m_1^S}{\psi_1}\left(1 - \mathrm{e}^{-\psi_1(t-s)}\right).$$

Therefore, for $s < t$, we have

$$\mathbb{Cov}\left(\lambda_s^{-1}, \lambda_t^{-1}\right) = \mathbb{E}\left[\lambda_t^{-1}\lambda_s^{-1}\right] - \mathbb{E}\left[\lambda_t^{-1}\right]\mathbb{E}\left[\lambda_s^{-1}\right]$$

$$= \mathrm{e}^{-\psi_1(t-s)}\mathbb{E}\left[\lambda_s^{-2}\right] + \frac{m_1^S}{\psi_1}\left(1 - \mathrm{e}^{-\psi_1(t-s)}\right)\mathbb{E}\left[\lambda_s^{-1}\right] - \mathbb{E}\left[\lambda_t^{-1}\right]\mathbb{E}\left[\lambda_s^{-1}\right],$$

where the remaining expectations are given by (5.6) and (5.7). Furthermore, when we set $S' = 0$ in equation (2.1), we retrieve the covariance results in [3, Theorem 2, page 317] upon substituting the results of Lemmas 5.1 and 5.2 to get an expression for equation (5.9) and subsequently subtracting the quantity $\mathbb{E}[\lambda_t^{-1}] \cdot \mathbb{E}[\lambda_s^{-1}]$.

6. Numerical results

To confirm that the simulation algorithms of Section 4 agree with the reciprocal moments derived in Section 5, we compare the first reciprocal moments given by the theory to Monte Carlo estimates. The results are given in Figure 3. The theory and the simulated values agree nicely. This example is also used to illustrate the speed benefits to the composition simulation method over the thinning method. In particular, Table 1 shows how long the two algorithms took to generate a fixed number of realizations of the extrinsic stress-release process. The difference in performance can be explained by (i) the fact that the composition method is easily vectorized while thinning is not, and (ii) the thinning algorithm is inherently inefficient as it intentionally generates too many points and discards a possibly large fraction of them.
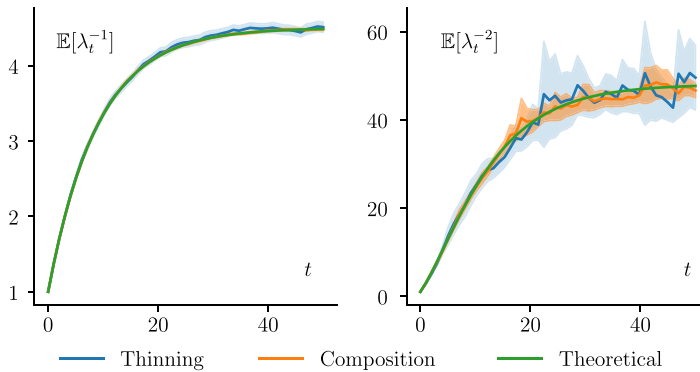
FIGURE 3. First reciprocal moments $\mathbb{E}\left[\lambda_t^{-1}\right]$ and second reciprocal moments $\mathbb{E}\left[\lambda_t^{-2}\right]$ for $t \in [0, 50]$ of an extrinsic stress-release process with $\lambda_0 = 1$, $\beta = 0.25$, $\rho = 1.25$, and $X_i \sim \mathsf{Exp}(3)$ and $Y_j \sim \mathsf{Exp}(10)$. The theoretical values, given by (5.6) and (5.7), are compared with crude Monte Carlo estimates using the two simulation methods from Section 4. Both simulation methods were allocated the same amount of computation time (55–60 seconds); as such, Algorithm 1.1 generated 375 000 sample paths of λ_t, whereas Algorithm 1.2 generated 25 000. The shaded regions indicate the 95% confidence intervals of the Monte Carlo estimates.

TABLE 1. Comparison of runtime (in seconds) to simulate a number of realizations of the extrinsic stress-release process until time $T = 100$ using Algorithms 1.1 and 1.2. The fastest time of three attempts is recorded. A grid search is performed to select the optimal step size $\Delta = 1.86$ for Algorithm 1.2 (this search is not included in the runtimes). The specification of the process (λ_0, β, ρ, etc.) is the same as in Figure 3.

Algorithm	Number of realizations			
	10^2	10^3	10^4	10^5
Composition	0.0205	0.0544	0.3600	3.2992
Thinning	0.3026	3.0329	30.761	310.32

7. Concluding remarks

In this article we have introduced a straightforward but computationally efficient way of simulating exactly for a class of generalized stress-release process. The idea stems from the observation that between contiguous jumps at points in time, the process satisfies the continuous part of the semimartingale and is thus governed by an ordinary differential equation. This permits us to derive an expression for the distribution between events. The end result is that we are able to sidestep the need to resort to thinning algorithms for the simulation of this class of point processes.

The explicit form of the infinitesimal generator for the extrinsic stress-release process is given. Theoretical reciprocal moments are derived which are then used to establish the validity of our simulation algorithms.

We envisage that the approach outlined in this paper extends naturally to the general coupled stress release models, i.e. the linked stress-release model [1, 2] with appropriate structures satisfying the martingale generators. Ongoing work is investigating such a problem.

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Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

Data

The Python code used to generate the numerical results is available at https://github.com/Pat-Laub/exact-simulation-of-extrinsic-stress-release-processes.

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